How to find the Laplace equation for droplets

Hello,

In this lecture I will show you how to find the Laplace equation so that you can calculate the internal pressure in droplets.

First, let us assume that we have a spherical droplet as in may occur in many real scenarios.

This sphere have a radius r and an internal pressure we can call P sub index [i,n]

If the internal pressure is positive, it applies a force pushing the liquid molecules outwardly

The droplet is made of a liquid whose more outer molecules are in contact with another material, in this case a gas. Therefore, there are two phases and only one interphase: Liquid-gas

So, if this droplet is meant to exist, the internal force pushing outwardly and the external forces pushing inwardly should be balanced otherwise the droplet would either burst or increase its internal pressure dramatically.

Given this the internal force, should be equal to two external inward forces: one due to the outside pressure. And the second one, which is due to the force exerted by the curved surface willing to return to the second dimension, and this depends on the material's surface tension [keep writing]

$$\mathrm{F}_{in} = F_{out} + F_{\gamma}$$

The internal force in this equation can be calculated as follows:

$Fin=P_{in}A_{sphere}$

Since we know how to calculate the area of a sphere, now the internal force equation can be rewritten as follows

$Fin= 4\pi r^2 p_{in}$.

The changes in surface area due to stretching causes the radius to increase from r to (r+dr), so this change can be calculated as follows

$$\mathrm{d}\sigma = \sigma_{final} - \sigma_{initial}$$

if this substitute in this equation the formula for calculating the area of a sphere we have

$$\mathrm{d}\sigma = 4\pi(\mathrm{r} + \mathrm{dr})^2 - 4\pi \mathrm{r}^2$$

by factoring the perfect square binomial we have

[on video]

We can eliminate this term for being very small or infinitesimal, and then we can have

$$d\sigma = 8\pi r dr$$

Now we need to calculate the work require to stretch the surface, let us use the following expression,

$$dw = \gamma(dr)$$

by substituting the change on total surface area we have

$$dw = \gamma(8\pi r dr)$$

Since work is the product of force and distance,

$$\text{Work} = \text{force} \cdot \text{distance}$$

We can isolate and calculate force as follows

[on video]

Remember the increase in radius due to stretching is very small.

[on video]

If we substitute in this expression the amount of work for stretching then we have

$$F_\gamma = 8\pi\gamma r$$

Since we already stated the forces should be balanced,

$$\text{Force}_{in} = \text{Force}_{out} + \text{Force}_\gamma$$

We can now substitute in this expression all the isolated equations of force and then obtain the following expression

$$4\pi r^2 p_{in} = 4\pi r^2 p_{out} + 8\pi\gamma r$$

We can cancel some factors by multiplying both sides of the equation by the same factor [on video]

And we finally obtain this expression

$$p_{in} = p_{out} + \frac{2\gamma}{r}$$

Than we can also write in the following way

[on video]

This is the Laplace equation for droplets.

This equation shows that a pressure increment in a curved surface is size related, i.e. the smaller the radius the larger the pressure increment. Remember, this mathematical deduction applies to a type of cavity for example a droplet (which can be a hole of vapour or gas filed with a bulk of liquid –there is only interface).